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9

TERMINOLOGY

addition theorems
angle sum and difference identities
compound angle formula
cosecant
cosine
cotangent
double angle formula
frequency
half-angle formula
products to sums and differences
Pythagorean identity
ratio
reciprocal function
reference angle
secant
sine
sum and difference formula
sums to products
tangent
trigonometric identity
unit circle
x-coordinate
y-coordinate

Prior learning

TRIGONOMETRY

TRIGONOMETRIC IDENTITIES

COMPOUND ANGLES

- prove and apply the angle sum, difference and double angle identities. (ACMSM044)

THE RECIPROCAL TRIGONOMETRIC FUNCTIONS, SECANT, COSECANT AND COTANGENT

- define the reciprocal trigonometric functions, sketch their graphs, and graph simple transformations of them. (ACMSM045)

TRIGONOMETRIC IDENTITIES

- prove and apply the Pythagorean identities (ACMSM046)
- prove and apply the identities for products of sines and cosines expressed as sums and differences (ACMSM047)
- convert sums $a \cos x + b \sin x$ to $R \cos(x \pm a)$ or $R \sin(x \pm a)$ and apply these to sketch graphs, solve equations of the form $a \cos x + b \sin x = c$ and solve problems (ACMSM048)
- prove and apply other trigonometric identities such as $\cos 3x = 4 \cos^3 x - 3 \cos x$. (ACMSM049) AC

9.01 RECIPROCAL TRIGONOMETRIC FUNCTIONS

Trigonometry is used in many fields of science, construction and engineering. Trigonometric functions are periodic functions that repeat themselves in a 'cycle'. Many functions that we see all around us are periodic. Periodic functions are found in the way waves move across water, light moves through space, sound moves through air and earthquakes travel around the Earth.

WS Basic trigonometry

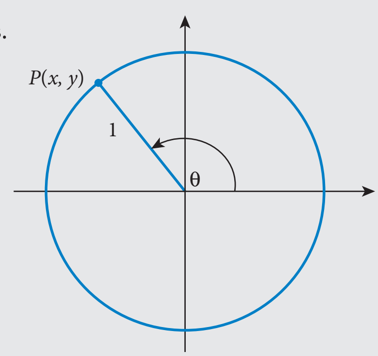
Previous work in trigonometric functions has dealt with the trigonometric ratios of sine, cosine and tangent.

Reciprocal trigonometric functions introduce three more functions: **secant**, **cosecant** and **cotangent**. They are the reciprocals of cosine, sine and tangent respectively.

IMPORTANT

Trigonometric functions

Angles are measured anticlockwise from the positive x-axis. For a point $P(x, y)$ at an angle of θ on the **unit circle**, we define each trigonometric **ratio** as below.



The **sine** of θ is the y-coordinate of the point P, so $\sin(\theta) = y$.

The **cosine** of θ is the x-coordinate of the point P, so $\cos(\theta) = x$.

The **tangent** of θ is the gradient of the line segment OP, so $\tan(\theta) = \dfrac{y}{x} = \dfrac{\sin(\theta)}{\cos(\theta)}$.

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The **secant** of θ is the reciprocal of the cosine, so sec (θ) = $\frac{1}{x} = \frac{1}{\cos(\theta)}$.

The **cosecant** of θ is the reciprocal of the sine, so cosec (θ) = $\frac{1}{y} = \frac{1}{\sin(\theta)}$.

The **cotangent** of θ is the reciprocal of the tangent, so cot (θ) = $\frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)}$.

The abbreviations of sine, cosine, tangent, secant, cosecant and cotangent are sin, cos, tan, sec, cosec and cot respectively.

The trigonometric ratios of angles are always abbreviated to the letters shown above.

IMPORTANT

Reciprocal trigonometric relationships are:

$$\text{cosec}(x) = \frac{1}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

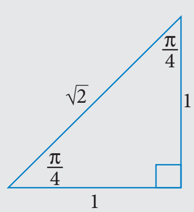
$$\cot(x) = \frac{1}{\tan(x)}$$

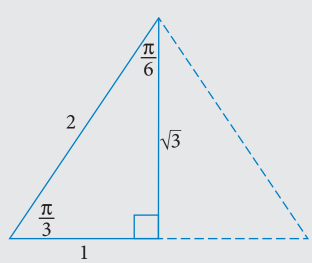
You should remember the exact values for the trigonometric functions of sin, cos and tan.

IMPORTANT

The exact values of the trig ratios for the angles $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°) and $\frac{\pi}{3}$ (60°), are shown in the table and **standard triangles** below.

Angle	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$





Example 1

State the exact values for the reciprocal trig functions: cot (30°), sec (45°) and cosec (60°).

Solution

Write each reciprocal function in terms of the original functions tan, cos or sin.

$$\cot(30°) = \frac{1}{\tan(30°)}$$
$$\sec(45°) = \frac{1}{\cos(45°)}$$
$$\csc(60°) = \frac{1}{\sin(60°)}$$

Substitute the exact values for tan (30°), cos (45°) and sin (60°).

$$\cot(30°) = \frac{1}{\frac{1}{\sqrt{3}}}$$
$$\sec(45°) = \frac{1}{\frac{1}{\sqrt{2}}}$$
$$\operatorname{cosec}(60°) = \frac{1}{\frac{\sqrt{3}}{2}}$$

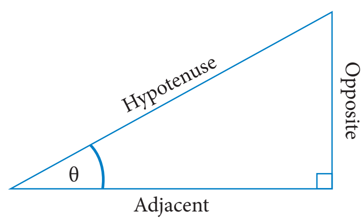
Simplify each fraction and write the answers.

$$\cot(30°) = \sqrt{3}$$
$$\sec(45°) = \sqrt{2}$$
$$\operatorname{cosec}(60°) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Degrees and radians can both be used to display symmetry.

Trigonometric ratios of triangle sides

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \operatorname{cosec}(\theta) = \frac{\text{Hypotenuse}}{\text{Opposite}}$$
$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \sec(\theta) = \frac{\text{Hypotenuse}}{\text{Adjacent}}$$
$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} \qquad \cot(\theta) = \frac{\text{Adjacent}}{\text{Opposite}}$$



Remember that cos (45) means that the 45 is a huge angle in radians, but cos (45°) means the angle is in degrees.

Degrees and radians

You know the exact values for some angles less than 90° (less than $\frac{\pi}{2}$). Values in other quadrants for secant, cosecant and cotangent follow the same patterns of symmetry that sine, cosine and tangent follow.

Whenever sine, cosine or tangents are positive or negative, so are their reciprocal pairs. This is summarised using the familiar CAST (or ASTC) diagram for a circle of radius r.

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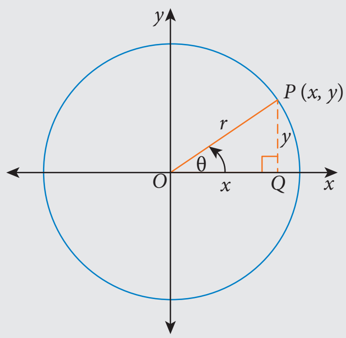
IMPORTANT

For any point $P(x, y)$ on a circle with radius r and centre at the origin:

$\sin(\theta) = \frac{y}{r}$, $\cos(\theta) = \frac{x}{r}$ and $\tan(\theta) = \frac{y}{x}$

$\text{cosec}(\theta) = \frac{r}{y}$, $\sec(\theta) = \frac{r}{x}$ and $\cot(\theta) = \frac{x}{y}$



The acute angle, θ in this case, is always identified as the angle in the 1st quadrant that is drawn between the terminal side of θ and the x-axis. This angle is known as the **reference angle**.

You know that in the 2nd quadrant:

$\sin(\pi - \theta) = \sin(\theta)$

$\cos(\pi - \theta) = -\cos(\theta)$

$\tan(\pi - \theta) = -\tan(\theta)$

It follows that for reciprocal functions in the 2nd quadrant:

$\text{cosec}(\pi - \theta) = \text{cosec}(\theta)$

$\sec(\pi - \theta) = -\sec(\theta)$

$\cot(\pi - \theta) = -\cot(\theta)$

IMPORTANT

Symmetry of the unit circle for reciprocal functions

2nd quadrant

$\text{cosec}(\pi - \theta) = \text{cosec}(\theta)$

$\sec(\pi - \theta) = -\sec(\theta)$

$\cot(\pi - \theta) = -\cot(\theta)$

3rd quadrant

$\text{cosec}(\pi + \theta) = -\text{cosec}(\theta)$

$\sec(\pi + \theta) = -\sec(\theta)$

$\cot(\pi + \theta) = \cot(\theta)$

4th quadrant

$\text{cosec}(2\pi - \theta) = -\text{cosec}(\theta) = \text{cosec}(-\theta)$

$\sec(2\pi - \theta) = \sec(\theta) = \sec(-\theta)$

$\cot(2\pi - \theta) = -\cot(\theta) = \cot(-\theta)$

IMPORTANT

Signs of the trig ratios

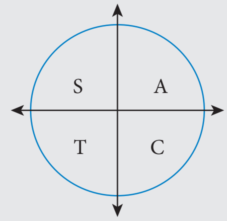
1st quadrant: All ratios are positive (**A**)

2nd quadrant: Sin only is positive (**S**)

3rd quadrant: Tan only is positive (**T**)

4th quadrant: Cos only is positive (**C**)

The above can be remembered by using the mnemonic '**CAST**' or '**A**ll **S**cience **T**eachers are **C**urious'.



Example 2

Simplify the expression $\cot\left(2\pi - \frac{\pi}{3}\right)$.

Solution

cot is the reciprocal of tan.

$$\cot\left(2\pi - \frac{\pi}{3}\right) = \frac{1}{\tan\left(2\pi - \frac{\pi}{3}\right)}$$

Note that $\cot\left(2\pi - \frac{\pi}{3}\right)$ places the angle in the fourth quadrant. tan and cot are both negative in the fourth quadrant.

$$\cot\left(2\pi - \frac{\pi}{3}\right) = -\cot\left(\frac{\pi}{3}\right)$$

Evaluate $\cot\left(\frac{\pi}{3}\right)$.

$$-\cot\left(\frac{\pi}{3}\right) = -\frac{1}{\tan\left(\frac{\pi}{3}\right)}$$

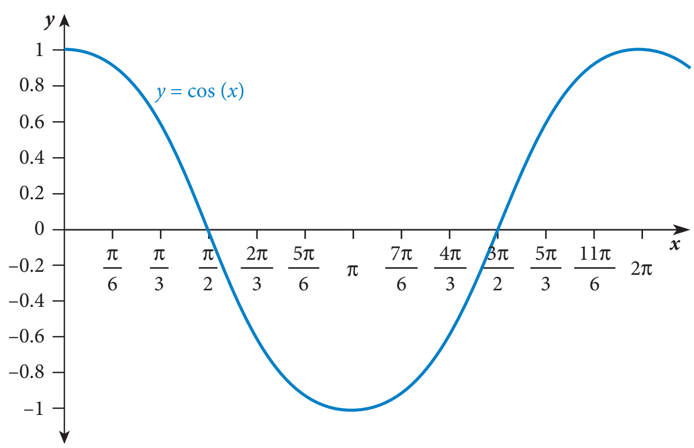
Write the answer.

$$\cot\left(2\pi - \frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$

WS Solving periodic functions

The graphs of secant, cosecant and cotangent can be developed from the graphs of cosine, sine and tangent.

The graph of $y = \cos(x)$ for $0 \le x \le 2\pi$ looks like:



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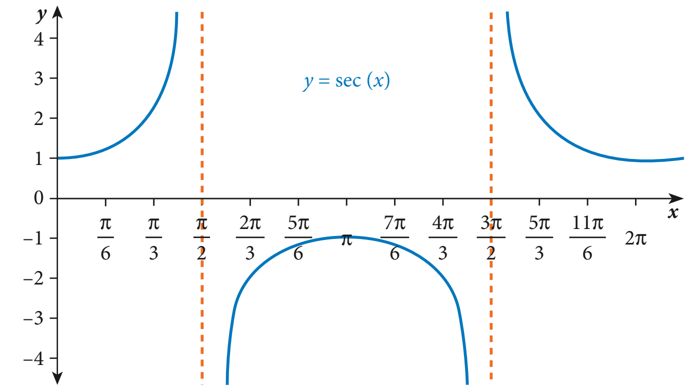
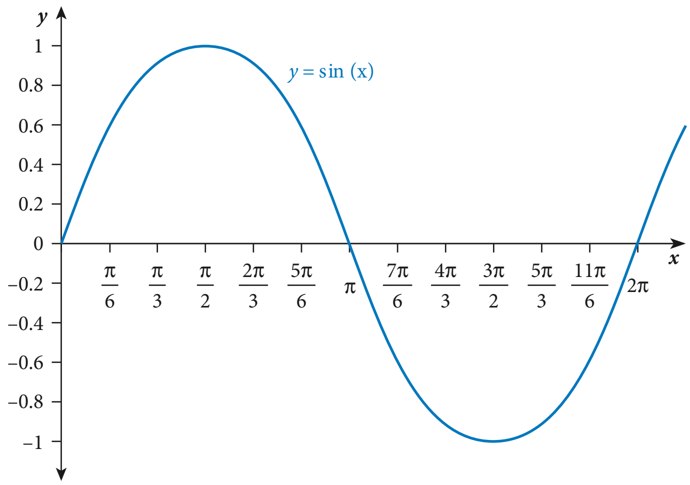
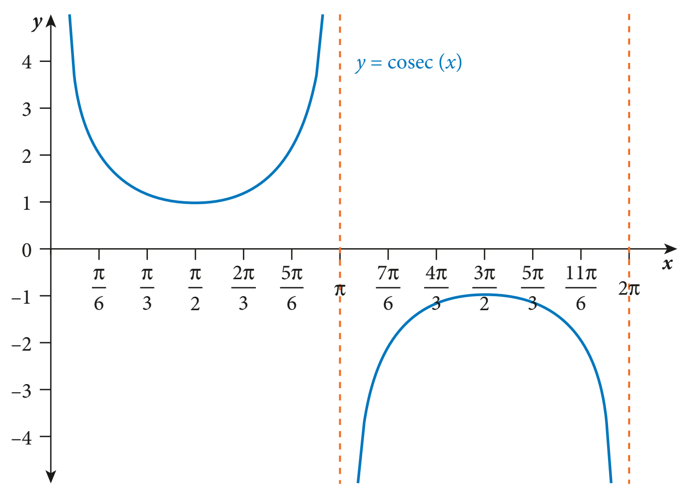
The graph of $y = \sec(x)$ for $0 \le x \le 2\pi$ looks like:



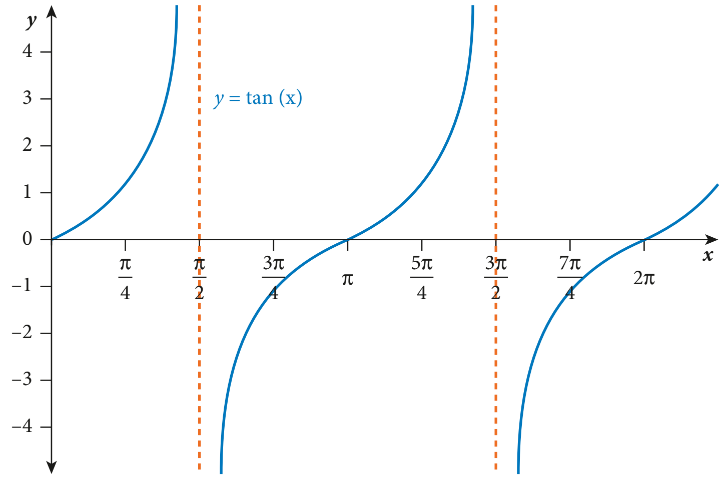
The graph of $y = \sin(x)$ for $0 \le x \le 2\pi$ looks like:



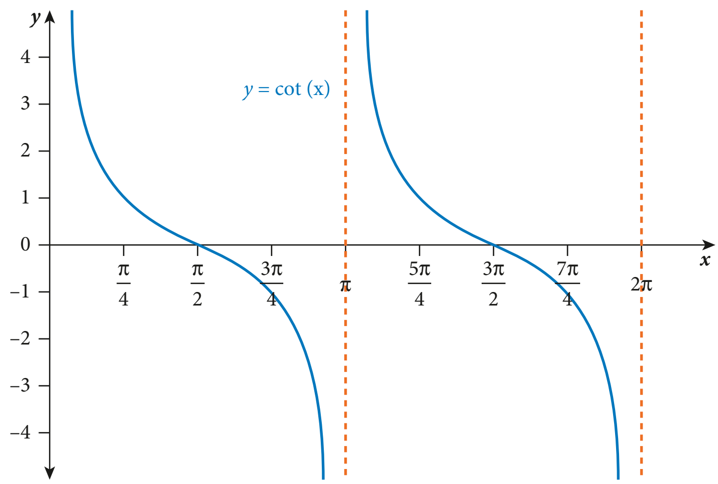
The graph of $y = \operatorname{cosec}(x)$ for $0 < x < 2\pi$ looks like:



The graph of $y = \tan(x)$ for $0 \le x \le 2\pi$ looks like:



The graph of $y = \cot(x)$ for $0 < x < 2\pi$ looks like:



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IMPORTANT

Graphs of reciprocal functions

In general, if $y = f(x)$, then its reciprocal $y = \frac{1}{f(x)}$ follows the rules:

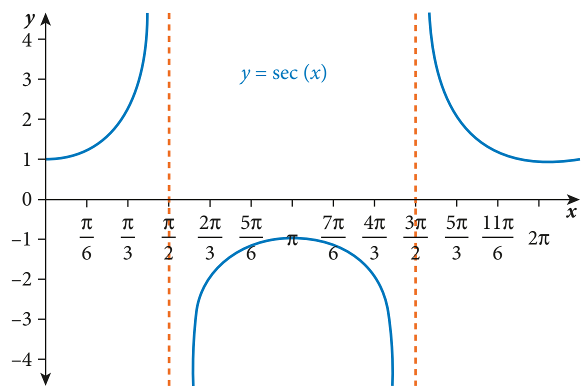
- if there are any x-intercepts in $y = f(x)$, then a vertical asymptote will be found at that point for the graph of $y = \frac{1}{f(x)}$ as you cannot divide by zero.
- if the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect, they will do so at the points $y = \pm 1$
- if the y values in $y = f(x)$ approach infinity as x approaches infinity, then the y values in $y = \frac{1}{f(x)}$ will approach zero as x approaches infinity.
- if $y \to 0$ in $y = \frac{1}{f(x)}$ as $x \to \infty$, then $y \to \infty$ in $y = f(x)$ as $x \to \infty$
- the sign of the reciprocal function will be the same as the sign of the function

Example 3

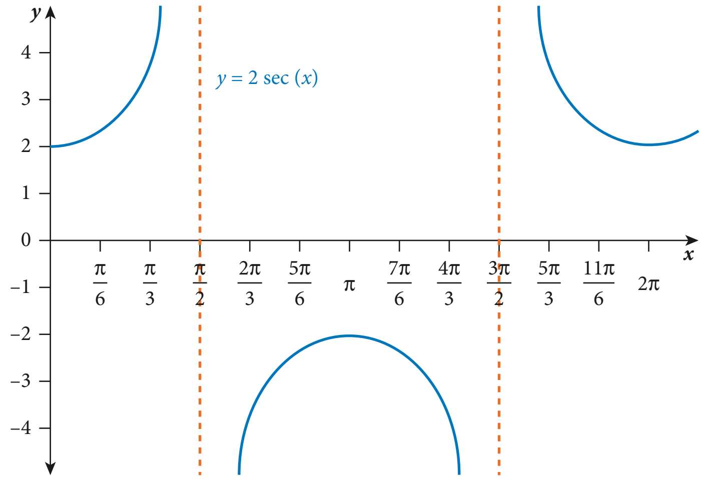
Sketch the graph of $y = 2 \sec (x) + 1$ for $0 \le x \le 2\pi$.

Solution

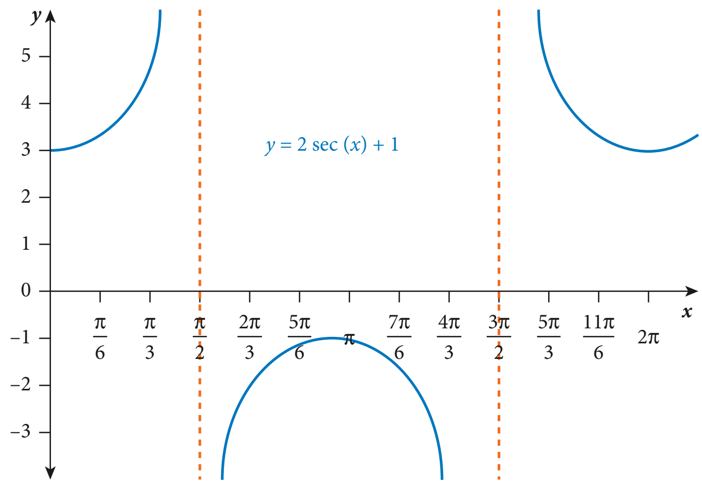
$\sec (x)$ is the reciprocal of $\cos (x)$ so its asymptotes are where $\cos (x) = 0$. This is at $\frac{\pi}{2}, \frac{3\pi}{2}$ and so on. Sketch the graph of $\sec (x)$ for $0 \le x \le 2\pi$.



Now draw a graph of $y = 2\ \sec\ (x)$ for $0 \leq x \leq 2\pi$, by multiplying all y values by 2.

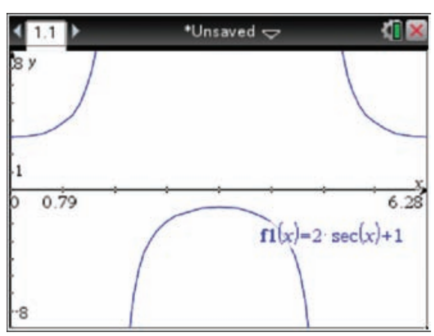


Now draw a graph of $y = 2\ \sec\ (x) + 1$ for $0 \leq x \leq 2\pi$, by shifting the previous graph upwards 1 unit.



TI-Nspire CAS

Insert a Graph page and make the Window settings $0 \leq x \leq 2$ with scale $\pi/4$ and $-8 \leq y \leq 8$ with scale 1. Press **µ** and put $\mathbf{f1}(x) =$ $2\ \sec\ (x) + 1$.



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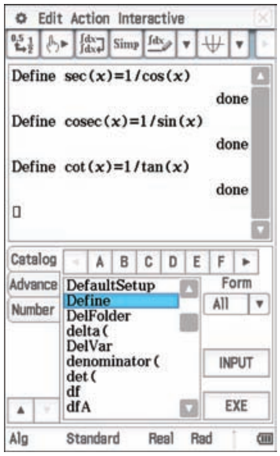
ClassPad

The ClassPad does not have specific defined functions for sec, cosec or cot, so the reciprocals of sin, cos and tan must be used. It is probably easiest to define these three functions using the Main $\sqrt{\alpha}$ application and **Define**, which is found in the [Catalog] menu (after pressing [Keyboard] and tapping [▼]). You will only need to do this once.

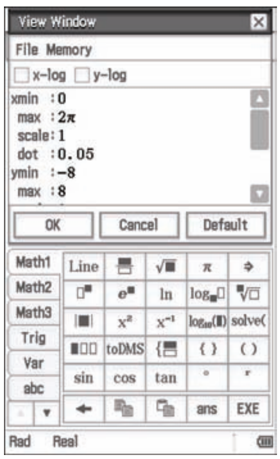
Make sure the calculator is set to radians (**Rad**).



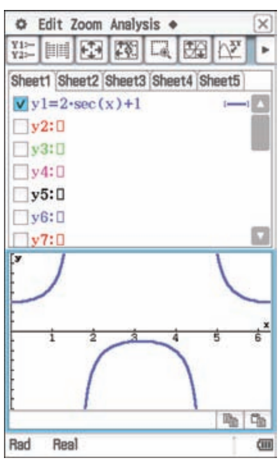
Use the **Graph&Table** application and set

y1=2sec(x)+1

Tap [View Window icon] for **View Window** to set $0 \le x \le 2\pi$ and $-8 \le y \le 8$. Tap **OK**.



Tap [graph icon] to draw the graph.



EXERCISE 9.01 Reciprocal trigonometric functions

Concepts and techniques

1 Example 1 $\cot\left(\frac{\pi}{3}\right) =$

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C 1 D $\sqrt{2}$ E $\sqrt{3}$

2 $\operatorname{cosec}\left(\frac{\pi}{4}\right) =$

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C 1 D $\sqrt{2}$ E $\sqrt{3}$

3 $\sec\left(\frac{\pi}{6}\right) =$

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C $\sqrt{3}$ D $\frac{2}{\sqrt{3}}$ E $\frac{\sqrt{3}}{2}$

4 cosec (90°) =

A 0 B $\frac{2}{\sqrt{3}}$ C $\frac{1}{\sqrt{2}}$ D 1 E undefined

5 cot (45°) =

A $\frac{1}{\sqrt{3}}$ B $\frac{1}{\sqrt{2}}$ C 1 D $\sqrt{2}$ E $\sqrt{3}$

6 Example 2 Evaluate:

a $\cot\left(\pi - \frac{\pi}{4}\right)$ b $\sec\left(\pi - \frac{\pi}{4}\right)$ c $\operatorname{cosec}\left(\pi + \frac{\pi}{3}\right)$ d $\sec\left(2\pi - \frac{\pi}{3}\right)$

e $\cot\left(\pi + \frac{\pi}{6}\right)$ f $\sec\left(\pi + \frac{\pi}{4}\right)$ g $\sec\left(2\pi + \frac{\pi}{4}\right)$ h $\cot\left(2\pi + \frac{\pi}{6}\right)$

7 Write a simplified expression for:

a $\sec(2\pi + \theta)$ b $\sec(\pi - \theta)$ c $\operatorname{cosec}(2\pi - \theta)$

d $\cot(\pi + \theta)$ e $\operatorname{cosec}(\pi + \theta)$ f $\cot(2\pi - \theta)$

8 Example 3 For $0 < x < 2\pi$, sketch the graph of $y = 3\operatorname{cosec}(x) - 1$

9 For $0 < x < 2\pi$, sketch the graph of $y = -\sec\left(x - \frac{\pi}{2}\right) + 1$

10 For $0 < x < 2\pi$, sketch the graph of $y = 4\cot(2x) + 1$

11 For $-\pi \le x \le \pi$, sketch the graph of $y = -2\sec\left(2x + \frac{\pi}{2}\right)$

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Reasoning and communication

12 Evaluate and compare $\sec\left(\frac{\pi}{4}\right)$ and $\operatorname{cosec}\left(\frac{\pi}{4}\right)$.

13 Evaluate and compare $\sec\left(\frac{\pi}{3}\right)$ and $\operatorname{cosec}\left(\frac{\pi}{6}\right)$.

14 Evaluate:

a $\sec\left(\frac{\pi}{6}\right)$

b $\cot\left(\frac{\pi}{4}\right)$

c $\operatorname{cosec}\left(-\frac{\pi}{3}\right)$

d $\operatorname{cosec}\left(\frac{2\pi}{3}\right)$

e $\sec(0)$

f $\cot(0)$

g $\cot\left(-\frac{4\pi}{3}\right)$

h $\sec\left(-\frac{7\pi}{4}\right)$

i $\operatorname{cosec}\left(\frac{3\pi}{2}\right)$

i $\sec\left(\frac{\pi}{3}\right)$

15 Use the graph of $y = \tan(x)$ to sketch $y = \cot(x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

16 Use the graph of $y = \cos(3x)$ to sketch the graph of $y = \sec(3x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

17 Use the graph of $y = \sin(4x)$ to sketch the graph of $y = \operatorname{cosec}(4x)$ from 0 to π.

18 Use the graph of $y = \tan(2x)$ to sketch the graph of $y = \cot(2x)$ from $-\pi$ to π.

19 Use the graph of $y=\cos\left(x-\frac{\pi}{4}\right)$ to sketch the graph of $y=\sec\left(x-\frac{\pi}{4}\right)$ from $-\pi$ to π.

20 Use the graph of $y=\sin\left(x+\frac{\pi}{6}\right)$ to sketch the graph of $y=\operatorname{cosec}\left(x+\frac{\pi}{6}\right)$ from 0 to 2π.

21 Use the graph of $y=\tan\left(x+\frac{\pi}{4}\right)$ to sketch the graph of $y=\cot\left(x+\frac{\pi}{4}\right)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

22 Sketch the graph of $y=\operatorname{cosec}\left(x+\frac{\pi}{4}\right)$ from 0 to 2π.

23 Sketch the graph of $y=\sec\left(3x+\frac{\pi}{4}\right)$ from 0 to π.

24 Sketch the graph of $y=\cot\left(2x-\frac{\pi}{4}\right)$ from 0 to π.

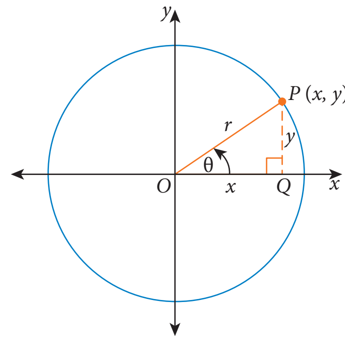
25 Sketch the graph of $y=\sec\left(2x-\frac{\pi}{3}\right)$ from 0 to π.

9.02 THE PYTHAGOREAN IDENTITIES

The earliest applications of trigonometry were in the fields of astronomy, navigation and surveying. Trigonometry was used to calculate distances that could not be measured directly. You may have investigated how to measure the height of tall structures like the school flagpole, weather vane, or wind turbine in your own previous work on trigonometry.

You should already be familiar with Pythagoras' theorem in a right angled triangle.

Pythagorean identities are so called because they follow the rules for Pythagoras' theorem.



For any point $P(x, y)$ on a circle of radius r with centre at the origin:

$$\sin(\theta) = \frac{y}{r}$$

$$\cos(\theta) = \frac{x}{r}$$

$$\tan(\theta) = \frac{y}{x}$$

Using Pythagoras' theorem, we get $r^2 = x^2 + y^2$.

Example 4

Prove the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$

Solution

Using Pythagoras' theorem, rewrite this relationship in terms of the lengths in the first quadrant of the unit circle.	Using the equation $c^2 = a^2 + b^2$, we get the relationship $r^2 = x^2 + y^2$
Relate the lengths r, x and y to trig functions.	We know that $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$
Rearrange the trig function equations.	$\cos(\theta) = \frac{x}{r} \Rightarrow x = r\cos(\theta)$ and $\sin(\theta) = \frac{y}{r} \Rightarrow y = r\sin(\theta)$
Substitute using $r^2 = x^2 + y^2$.	We know that $r^2 = x^2 + y^2$, giving $r^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta)$ So for any r: $1 = \cos^2(\theta) + \sin^2(\theta)$ **QED**

This proves the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$.

Rearrangement and substitution of A for θ gives the following.

IMPORTANT

The basic Pythagorean identity

$\sin^2(A) + \cos^2(A) = 1$

$\cos^2(A) = 1 - \sin^2(A)$

$\sin^2(A) = 1 - \cos^2(A)$

where A is any angle, measured in either degrees or radians.

Example 5

Prove the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$

Solution

Write the basic Pythagorean identity. $\sin^2(\theta) + \cos^2(\theta) = 1$

Divide by $\cos^2(\theta)$.

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

Simplify and rearrange.

$$\left[\frac{\sin(\theta)}{\cos(\theta)}\right]^2 + 1 = \left[\frac{1}{\cos(\theta)}\right]^2$$

Write the basic functions. $\tan^2(\theta) + 1 = \sec^2(\theta)$ **QED**

Using Pythagoras' theorem

You can prove the other identity in the same way.

IMPORTANT

Pythagorean identities

$\cos^2(A) + \sin^2(A) = 1$

$\tan^2(A) + 1 = \sec^2(A)$

$\cot^2(A) + 1 = \mathrm{cosec}^2(A)$

where A is any angle, measured in either degrees or radians.

The triple bar symbol '≡' is often used in identities, meaning that the statement is true for every value of the unknown. For example, you could write $\cos^2(\theta) + \sin^2(\theta) \equiv 1$.

IMPORTANT

You can use the following hints in proving trigonometric identities.

1. Try starting with the most complicated-looking side of the identity.
2. Fractional expressions are generally more complicated than sums and differences. Products are the simplest.
3. If the expressions involve squares, try using the Pythagorean identities.
4. If you cannot see how to proceed, try changing all the functions to sin and cos.
5. If you are getting nowhere, try starting from the other side of the identity.

Example 6

Prove the identity $\text{cosec}^2(x) = \sec(x)\,\text{cosec}(x)\cot(x)$

Solution

The RHS looks more complicated. $\text{RHS} = \sec(x)\,\text{cosec}(x)\cot(x)$

Change to simpler trig functions.

$$= \frac{1}{\cos(x)} \times \frac{1}{\sin(x)} \times \frac{1}{\tan(x)}$$

Use $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

$$= \frac{1}{\cos(x)} \times \frac{1}{\sin(x)} \times \frac{1}{\frac{\sin(x)}{\cos(x)}}$$

Simplify and cancel.

$$= \frac{1}{\cos(x)} \times \frac{1}{\sin(x)} \times \frac{\cos(x)}{\sin(x)}$$

$$= \frac{1}{\sin(x)} \times \frac{1}{\sin(x)}$$

Simplify.

$$= \left[\frac{1}{\sin(x)}\right]^2$$

$$= \text{cosec}^2(x)$$

$$= \text{LHS}$$

QED

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EXERCISE 9.02 The Pythagorean identities

Concepts and techniques

1 Example 4 Prove that $\sin^2(A) + \cos^2(A) \equiv 1$ from the definitions.

2 Example 5 Prove that $\tan(A) \equiv \dfrac{\sin(A)}{\cos(A)}$ from the definitions.

3 Example 6 Prove that $1 + \cot^2(A) \equiv \text{cosec}^2(A)$.

4 Prove that $\cot(A) \equiv \dfrac{\cos(A)}{\sin(A)}$ from the definitions.

5 Show that $\tan\left(\dfrac{2\pi}{3}\right) = \dfrac{\sin\left(\frac{2\pi}{3}\right)}{\cos\left(\frac{2\pi}{3}\right)}$.

Reasoning and communication

6 Prove that $[1 - \sin(\theta)][1 + \sin(\theta)] = \cos^2(\theta)$.

7 Prove that $\sin^4(\theta) - \cos^4(\theta) = 1 - 2\cos^2(\theta)$.

8 Prove that $\left[\cot(x) + \text{cosec}(x)\right]^2 = \dfrac{1 + \cos(x)}{1 - \cos(x)}$.

9 Prove that $\sec(y) + \tan(y) = \dfrac{1 + \sin(y)}{\cos(y)}$.

10 Prove that $2\,\text{cosec}^2(a) = \dfrac{1}{1 - \cos(a)} + \dfrac{1}{1 + \cos(a)}$.

11 Prove that $\dfrac{\sin(\beta) + \cos(\beta)}{\cos(\beta)\sin(\beta)} = \sin(\beta)\left[1 + \tan(\beta)\right] + \cos(\beta)\left[1 + \cot(\beta)\right]$.

12 Prove that $1 + \sin(\theta)\cos(\theta) = \dfrac{\sin(\theta)\cos(\theta)\left[\cos(\theta)\cot(\theta) - \sin(\theta)\tan(\theta)\right]}{\cos(\theta) - \sin(\theta)}$.

9.03 ANGLE SUM AND DIFFERENCE IDENTITIES

You can simplify problems that include more complex angles using the **angle sum and difference identities**. These are also commonly called '**sum and difference formulas**' or '**compound angle formulas**'.

We usually use x, y, etc. for the variables because they represent real numbers.

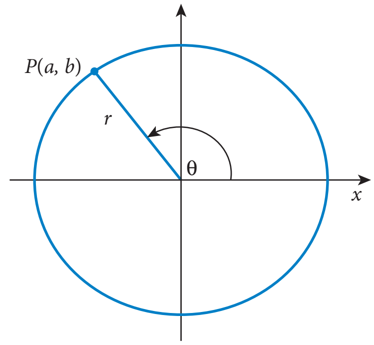
The identity

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

is proved first and the others are developed from it.

Create a circle diagram, with radius r, where the angle $(x - y)$ can be represented.

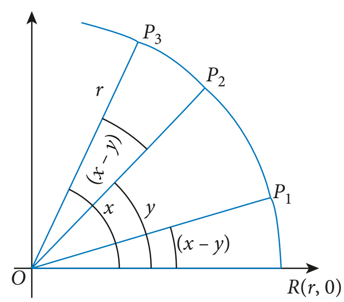
Choose a point $p(a, b)$ at a radius of r on the ray OP, drawn at an angle of θ with the x-axis.



We know that $\sin(\theta) = \frac{b}{r}$ and $\cos(\theta) = \frac{a}{r}$ so $b = r \sin(\theta)$ and $a = r \cos(\theta)$.
Point p at radius r on the ray OP drawn at an angle of θ with the x-axis can be written as $P(r \cos(\theta), r \sin(\theta))$.

Focus on just the first quadrant of the circle, plotting points on the arc that are subtended by the angle $(x - y)$.

Consider points P_1, P_2 and P_3 drawn at radius r in rays OP_1, OP_2 and OP_3 at angles $(x - y)$, y, x with the x-axis.



These points can be written as $P_1(r \cos(x - y), r \sin(x - y))$, $P_2(r \cos(y), r \sin(y))$ and $P_3(r \cos(x), r \sin(x))$.
$R(r, 0)$ is on the x-axis at radius r.

The angle subtended by the chord P_2P_3 at the centre of the circle is the angle x. This is because the chord P_2P_3 subtends the angle $(x + y) - y$, which equals x.

The chords OP_1, OP_2 and OP_3 are of equal length as they are radii of the circle with radius r.
Since the chords P_2P_3 and RP_1 subtend the same angle, x, at the centre of the circle, they must be equal in length.
Thus $|P_2P_3| = |RP_1|$

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Continue the proof by using the distance formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.	Using the distance formula, we get $\sqrt{[r\cos(x)-r\cos(y)]^2+[r\sin(x)-r\sin(y)]^2}$ $=\sqrt{[r-r\cos(x-y)]^2+[0-r\sin(x-y)]^2}$
Square both sides.	$[r\cos(x)-r\cos(y)]^2+[r\sin(x)-r\sin(y)]^2$ $=[r-r\cos(x-y)]^2+[0-r\sin(x-y)]^2$
Take out the common factor of r^2 and then divide both sides by r^2.	$[\cos(x)-\cos(y)]^2+[\sin(x)-\sin(y)]^2$ $=[1-\cos(x-y)]^2+[-\sin(x-y)]^2$
Multiply out the brackets and simplify, using the Pythagorean identity. Collect and rearrange the terms.	$\cos^2(x)-2\cos(x)\cos(y)+\cos^2(y)+\sin^2(x)-2\sin(x)\sin(y)+\sin^2(y)$ $=1-2\cos(x-y)+\cos^2(x-y)+\sin^2(x-y)$ This gives $2-2\cos(x)\cos(y)-2\sin(x)\sin(y)$ $=2-2\cos(x-y)$ leading to $-\cos(x)\cos(y)-\sin(x)\sin(y)=-\cos(x-y)$
Simplify to get the result required.	$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$ **QED**

The identity $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$ can also be written as $\cos(A-B)\equiv\cos(A)\cos(B)+\sin(A)\sin(B)$.

IMPORTANT

Angle sum and difference identities

The following identities apply for all A and B.

$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$

$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)$

$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$

$\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$

$$\tan(A+B)=\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

$$\tan(A-B)=\frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

You will notice that the sum and difference formulas for tangent look quite different from the sine and cosine formulas. It is quite common for tangent expressions and graphs to be quite different from those of sine and cosine. This is because it is a quotient.

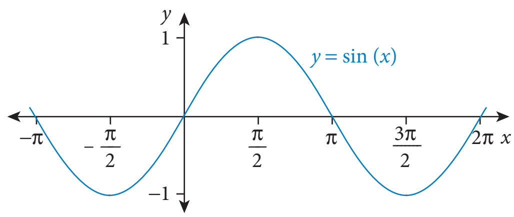
Example 7

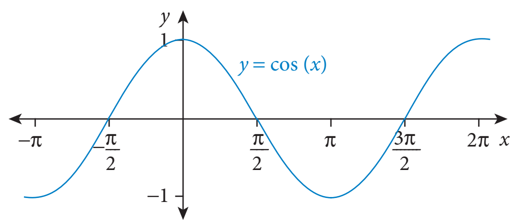
Expand cos $(3x + 2y)$.

Solution

Write the expression.	$\cos(3x + 2y)$
Use cos $(A + B)$	$= \cos(3x)\cos(2y) - \sin(3x)\sin(2y)$
Write the result.	$\cos(3x + 2y) = \cos(3x)\cos(2y) - \sin(3x)\sin(2y)$

You may have already learnt about **complementary** trig functions.





The graphs of $y = \sin(x)$ and $y = \cos(x)$ look very similar. In fact, in the graph of $y = \cos(x)$, if you use a **phase shift** of $\frac{\pi}{2}$ in the positive direction of the x-axis, you will notice that the cosine graph looks exactly like the sine graph.

WS
Simplifying periodic functions

There are complementary relationships between sine and cosine, tangent and cotangent, secant and cosecant.

Note the prefix 'co'.

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Important

Complementary function identities

The following identities apply for all values of θ.

$$\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos(\theta)$$

$$\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin(\theta)$$

$$\tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot(\theta)$$

Also

$$\sin\left(\frac{3\pi}{2} \pm \theta\right) = -\cos(\theta)$$

$$\cos\left(\frac{3\pi}{2} \pm \theta\right) = \pm\sin(\theta)$$

$$\tan\left(\frac{3\pi}{2} \pm \theta\right) = \mp\cot(\theta)$$

Example 8

Use the expansion of cos $(y - x)$ to simplify $\cos\left(\frac{\pi}{2} - x\right)$.

Solution

Write the theorem for cos $(y - x)$.

$$\cos(y - x) = \cos(y)\cos(x) + \sin(y)\sin(x)$$

Let $y = \frac{\pi}{2}$.

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2}\right)\cos(x) + \sin\left(\frac{\pi}{2}\right)\sin(x)$$

Evaluate.

$$= 0 \times \cos(x) + 1 \times \sin(x)$$

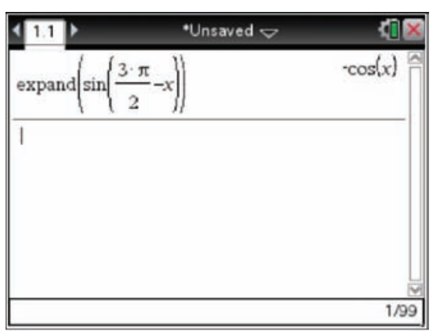
Write the result.

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$

Complementary angle properties

TI-Nspire CAS

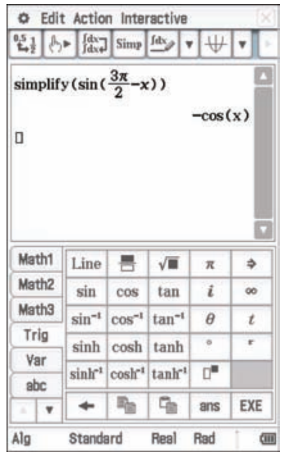
Make sure that you have the Angle in radians and the Calculation Mode in Auto or Exact.
Use the expand() instruction, either from the catalogue (k)m or from b, 3: Algebra and 3: Expand.



ClassPad

From the Main $\sqrt{\alpha}$ application you can use **simplify** (first tap **Action** then **Transformation**) to simplify these expressions.

Make sure that the calculator is set to radians (**Rad**).



You can use the sum and difference formulas to prove identities in a similar way to those already encountered.

Example 9

Prove that $\cot(x)+\cot(y)=\dfrac{\sin(x+y)}{\sin(x)\sin(y)}$

Solution

The right-hand side looks more complicated, so start with it.

$$\text{RHS}=\frac{\sin(x+y)}{\sin(x)\sin(y)}$$

Use the theorem for sin $(x + y)$.

$$=\frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\sin(x)\sin(y)}$$

Express this as separate fractions.

$$=\frac{\sin(x)\cos(y)}{\sin(x)\sin(y)}+\frac{\cos(x)\sin(y)}{\sin(x)\sin(y)}$$

$$=\frac{\cos(y)}{\sin(y)}+\frac{\cos(x)}{\sin(x)}$$

Use the definition for cot and rearrange.

$$= \cot(y) + \cot(x)$$

$$= \cot(x) + \cot(y)$$

$$= \text{LHS}$$

QED

EXERCISE 9.03 Angle sum and difference identities

Concepts and techniques

1 Example 7 Expand the following.

a sin $(2x + y)$ b cos $(4x - 3y)$ c sin $(3p - 5q)$

d tan $(2e - f)$ e cos $(3\alpha + 2\beta)$ f sin $(4m + n)$

g tan $(w + 3x)$ h sin $(2D - 5E)$ i cos $(P - 4Q)$

2 Example 8 Simplify the following.

a $\sin(\pi - x)$ b $\cos(\pi + x)$ c $\tan\left(\frac{\pi}{2} - x\right)$

d $\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$ e $\cot\left(\frac{3\pi}{2} - x\right)$ f $\sec\left(x + \frac{\pi}{2}\right)$

g $\sin(2\pi - x)$ h $\cos\left(\frac{3\pi}{2} - x\right)$ i $\tan(x - \pi)$

3 Use the expansion of $\cos(x - y)$ and $\sin(x - y)$ to prove that $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$ and $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$.

4 Uses expansions of $\sin\left(\frac{\pi}{2} + x\right)$ and $\cos\left(\frac{\pi}{2} + x\right)$ to prove that $\tan\left(\frac{\pi}{2} + x\right) = -\cot(x)$.

Reasoning and communication

5 Prove that $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

6 Prove that $\tan(x - y) = \dfrac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$.

7 Example 9 Prove that $\cot(x + y) = \dfrac{\cot(x)\cot(y) - 1}{\cot(y) + \cot(x)}$.

8 Prove that $\tan(x) + \tan(y) = \dfrac{\sec(x)\sec(y)}{\operatorname{cosec}(x + y)}$.

9 Prove that $\sin(x + y) + \cos(x - y) = [\sin(x) + \cos(x)][\sin(y) + \cos(y)]$.

10 Prove that $\sin(x + y)\sin(x - y) = \sin^2(x) - \sin^2(y)$.

11 Prove that $\dfrac{\cot(x) - \cot(y)}{1 - \cot(x)\cot(y)} = \dfrac{\sin(x - y)}{\cos(x + y)}$.

12 Prove that $\tan(x - y) + \tan(y - z) + \tan(z - x) = \tan(x - y)\tan(y - z)\tan(z - x)$.

9.04 DOUBLE ANGLE FORMULAS

Double angle formulas can be developed from sum and difference identities.

Example 10

Show that $\sin(2x) = 2\sin(x)\cos(x)$

Solution

Express $2x$ as $x + x$.	$\sin(2x) = \sin(x + x)$
Use $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.	$\sin(x + x) = \sin(x)\cos(x) + \cos(x)\sin(x)$
Collect terms.	$\sin(x + x) = 2\sin(x)\cos(x)$
Write the result.	$\sin(2x) = 2\sin(x)\cos(x)$

You can develop each of the double angle formulas in a similar way.

IMPORTANT

Double angle formulas

The following identities apply for all A.

$\sin(2A) = 2\sin(A)\cos(A)$

$\cos(2A) = \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)$

$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$

You have three choices for the double angle formula for $\cos(2A)$. You choose whichever is more suitable for the problem.

You can use the double angle formulas for multiple angles.

Example 11

Use the double angle formula for tan to find an expression for $\tan(6x)$.

Solution

Express $6x$ as $3x + 3x$.

$\tan(6x) = \tan(3x + 3x)$

Use the identity $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$ to find an expression for $\tan(6x)$.

$$\tan(3x+3x) = \frac{\tan(3x)+\tan(3x)}{1-\tan(3x)\tan(3x)}$$

Collect terms.

$$\tan(3x+3x) = \frac{2\tan(3x)}{1-\tan^2(3x)}$$

Write the result.

$$\tan(6x) = \frac{2\tan(3x)}{1-\tan^2(3x)}$$

It is important for you to remember that it is the **angle** that is doubled and not the rest of the expression.

You could continue to simplify questions similar to Example **11**, using a combination of double angle formulas and sum and difference identities.

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Example 12

Find an expression for sin (3x) in terms of sin (x) only.

Solution

Express 3x as 2x + x.	$\sin(3x) = \sin(2x + x)$
Use $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$= \sin(2x)\cos(x) + \cos(2x)\sin(x)$
Use the sin (2x) and cos (2x) formulas.	$= 2\sin(x)\cos(x)\cos(x) + [1 - 2\sin^2(x)]\sin(x)$
Remove the brackets.	$= 2\sin(x)\cos^2(x) + \sin(x) - 2\sin^3(x)$
Use $\sin^2(x) + \cos^2(x) = 1$	$= 2\sin(x)[1 - \sin^2(x)] + \sin(x) - 2\sin^3(x)$
Remove the brackets.	$= 2\sin(x) - 2\sin^3(x) + \sin(x) - 2\sin^3(x)$
Collect like terms.	$= 3\sin(x) - 4\sin^3(x)$
Write the result.	$\sin(3x) = 3\sin(x) - 4\sin^3(x)$

WS Advanced exponential and periodic functions 1

You can also use double angle formulas to prove trig identities as you did in Section **9.03**.

EXERCISE 9.04 Double angle formulas

Concepts and techniques

1 Example 10 Show that $\cos(2A) = 1 - 2\sin^2(A)$

2 Show that $2\cos^2(A) - 1 = 1 - 2\sin^2(A)$

3 Show that $\cos^2(y) - \sin^2(y) = 1 - 2\sin^2(y)$

4 Example 11 Show that $\sin(8A) = 2\sin(4A)\cos(4A)$.

5 Show that $\cos(5A) = 1 - 2\sin^2\left(\frac{5A}{2}\right)$

6 Use a suitable double angle formula to expand:
  a sin (4x) **b** cos (6A) **c** tan (4y) **d** cos (y) **e** tan (4x)

7 Example 12 Use sum and difference identities and double angle formulas, where suitable, to continue to expand the expressions in question **6a**, **b**, **c**.

Reasoning and communication

8 Prove that $\frac{1}{\cos(\theta) + \sin(\theta)} + \frac{1}{\cos(\theta) - \sin(\theta)} = 2\cos(\theta)\sec(2\theta)$

9 Prove that $\tan(4x) - \tan(x) = \frac{\sec(4x)\sec(x)}{\operatorname{cosec}(3x)}$.

10 Show that $\sin(4x) = 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x)$.

11 Show that $\cot(2x)=\dfrac{\cot^2(x)-1}{2\cot(x)}$.

12 Prove that $\cot(\theta)-\tan(\theta)=2\cot(2\theta)$.

13 Prove that $\tan(2x)=\dfrac{1}{1-\tan(x)}-\dfrac{1}{1+\tan(x)}$.

14 Prove that $\tan(x)=\cot(x)-2\cot(2x)$.

9.05 FINDING AND USING EXACT VALUES

You can simplify many expressions that include angles with exact trigonometric ratios using the sum and difference identities and the double angle identities. They also allow you to calculate the exact values of other angles, such as $\frac{\pi}{12}$.

Example 13

Find the exact value of cos (15°).

Solution

Think of 2 values, using 30°, 45° or 60° that will combine, using + or – to get 15°.	$60° - 45° = 15°$
Use the difference formula for cos.	$\cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$
Use this difference expression in cos (15°).	$\cos(60°-45°)=\cos(60°)\cos(45°)+\sin(60°)\sin(45°)$
Use exact values to evaluate the RHS of the expression.	$=\frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}$ $=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$ $=\frac{\sqrt{2}+\sqrt{6}}{4}$
Simplify the LHS.	$\cos(60°-45°)=\cos(15°)$
Write the answer.	$\cos(15°)=\frac{\sqrt{2}+\sqrt{6}}{4}$

You can use any of the other sum and difference identities to simplify angle expressions that can be expressed in terms of the three exact-value angles $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ (30°, 45°, 60°).

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Example 14

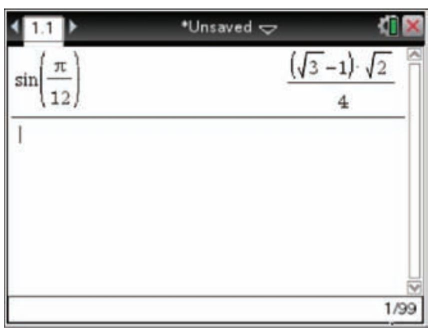
Find the exact value of $\sin\left(\frac{\pi}{12}\right)$.

Solution

Think of 2 values, using $\frac{\pi}{6}, \frac{\pi}{4}$ or $\frac{\pi}{3}$ that will combine, using + or – to get $\frac{\pi}{12}$.	$\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$
Use the difference formula for sin.	$\sin(A-B)=\sin(A)\cos(B)-\cos(A)\sin(B)$
Use this difference expression in $\sin\left(\frac{\pi}{12}\right)$.	$\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$
Use exact values to evaluate the RHS of the expression.	$\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$ $=\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}-\frac{1}{2}\times\frac{\sqrt{2}}{2}$ $=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$ $=\frac{\sqrt{6}-\sqrt{2}}{4}$
Simplify the LHS.	$\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{12}\right)$

TI-Nspire CAS

Make sure that your calculator is set on radians and is in auto or exact Calculation Mode.

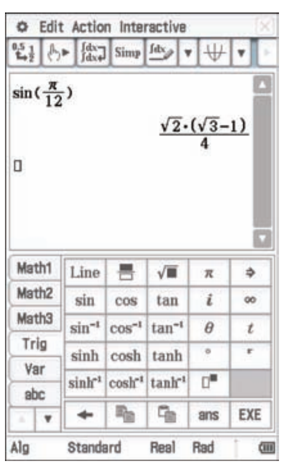


ClassPad

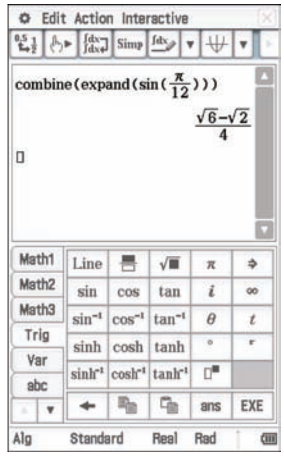
You can find exact values in one step.

Use the Main $\sqrt{\alpha}$ application and make sure that the calculator is set to **Standard**.

Sin, cos and tan and and fractions [fraction key] can be found on the Trig menu that is available after pressing Keyboard.



You can use the command **expand** to get rid of the brackets and **combine** to present the answer with a common denominator. (Both commands are on the menu accessed by tapping **Action** then **Transformation**.)



Write the answer. $\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$

Double angle formulas can be used to find the exact values of trigonometric ratios that have angles other than $\frac{\pi}{6}, \frac{\pi}{4}$ or $\frac{\pi}{3}$.

Example 15

Find the exact value of $\sin\left(\frac{3\pi}{8}\right)$ using a double angle formula.

Solution

Express $\frac{3\pi}{8}$ in terms of an angle with an exact trig value. $2\times\frac{3\pi}{8}=\frac{3\pi}{4}$

Write the cos $(2x)$ rule with sin (x). $\cos(2x) = 1 - 2\sin^2(x)$

Use the angle expression. $\cos\left(\frac{3\pi}{4}\right) = 1 - 2\sin^2\left(\frac{3\pi}{8}\right)$

Rearrange. $2\sin^2\left(\frac{3\pi}{8}\right) = 1-\cos\left(\frac{3\pi}{4}\right)$

Use the value $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ $\quad 2\sin^2\left(\frac{3\pi}{8}\right) = 1 + \frac{\sqrt{2}}{2}$

Simplify the RHS. $2\sin^2\left(\frac{3\pi}{8}\right) = \frac{2+\sqrt{2}}{2}$

Divide by 2. $\sin^2\left(\frac{3\pi}{8}\right) = \frac{2+\sqrt{2}}{4}$

Find the square root. $\sin\left(\frac{3\pi}{8}\right) = \pm\frac{\sqrt{2+\sqrt{2}}}{2}$

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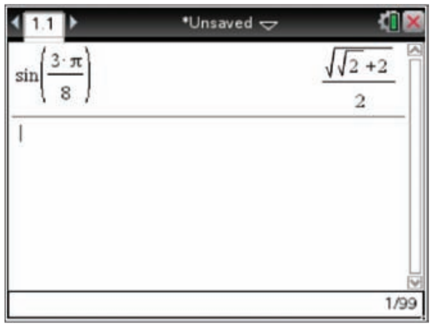
In the first quadrant sin (x) is positive. $\sin\left(\frac{3\pi}{8}\right)=\frac{\sqrt{2+\sqrt{2}}}{2}$

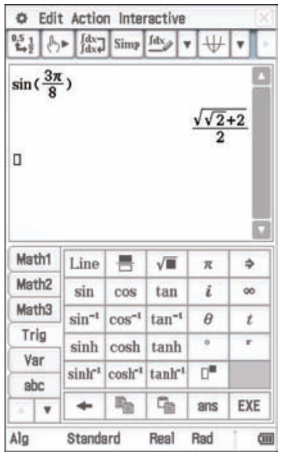
TI-Nspire CAS

You can check this on your CAS calculator. Make sure that it is set to radians and auto or exact Calculation Mode.



ClassPad

Make sure that your calculator is set on **Standard** calculation mode and radians (**Rad**).



EXERCISE 9.05 Finding and using exact values

Concepts and techniques

1 Example 13 Find the exact values of each of the following using angle sum and difference identities. Check your answers with your CAS calculator.

a tan (15°)　　b cos (75°)　　c $\tan\left(\frac{7\pi}{12}\right)$

d $\sin\left(\frac{5\pi}{12}\right)$　　e $\cot\left(\frac{11\pi}{12}\right)$　　f sin (105°)

g cos (195°)　　h $\sin\left(\frac{7\pi}{12}\right)$　　i sin (195°)

2 Example 14 Find the exact value of tan (75°), using angle sum and difference identities and a combination of the angles 45° and 30°.

3 Example 15 Find exact values of the following using a suitable double angle formula.

a $\sin\left(\frac{\pi}{12}\right)$ b $\cos\left(\frac{\pi}{8}\right)$ c $\tan\left(\frac{5\pi}{12}\right)$

d $\sin\left(\frac{\pi}{8}\right)$ e $\cos\left(\frac{5\pi}{12}\right)$ f $\tan\left(\frac{\pi}{8}\right)$

9.06 PRODUCTS TO SUMS

To simplify calculations, sometimes you will need to change products involving trigonometric functions into sums. The rules for these conversions can be proved using the addition theorems.

We first show that $\cos(A)\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Example 16

Show that $\cos(A)\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Solution

Simplify the RHS using sum and difference formulas.	**Proof** $\frac{1}{2}[\cos(A-B)+\cos(A+B)]$ $=\frac{1}{2}[\cos(A)\cos(B)+\sin(A)\sin(B)+\cos(A)\cos(B)-\sin(A)\sin(B)]$
Collect terms and simplify.	$=\frac{1}{2}\times 2\cos(A)\cos(B)$ $=\cos(A)\cos(B)$
Write the result.	$\cos(A)\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$ **QED**

Sum and difference identities similar to the one shown above yield the following rules for conversion from **products to sums and differences**.

IMPORTANT

Products as sums and differences

$\cos(A)\cos(B)=\frac{1}{2}[\cos(A-B)+\cos(A+B)]$

$\sin(A)\sin(B)=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$

$\sin(A)\cos(B)=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$

$\cos(A)\sin(B)=\frac{1}{2}[\sin(A+B)-\sin(A-B)]$

You will find these formulas useful when rearranging exact angle expressions.

Example 17

Given that sin (50°) ≈ 0.76604, use an appropriate products as sums and differences formula to evaluate the expression cos (40°) sin (10°). Give your answer correct to 3 decimal places.

Solution

Use the formula

$\cos(A)\sin(B)=\frac{1}{2}[\sin(A+B)-\sin(A-B)]$.

$\cos(40°)\sin(10°)$

$=\frac{1}{2}[\sin(40°+10°)-\sin(40°-10°)]$

Collect terms and evaluate.

$=\frac{1}{2}[\sin(50°)-\sin(30°)]$

$=\frac{1}{2}(0.76604-0.5)$

$=\frac{1}{2}\times 0.26604$

Write the result.

$=0.133$

Example 18

Express sin (4*x*) sin (7*x*) as a sum or a difference of trigonometric functions.

Solution

Use

$\sin(A)\sin(B)=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$.

$\sin(4x)\sin(7x)$

$=\frac{1}{2}[\cos(4x-7x)-\cos(4x+7x)]$

Collect terms and simplify.

$=\frac{1}{2}[\cos(-3x)-\cos(11x)]$

Remember that cos (−θ) = cos (θ).

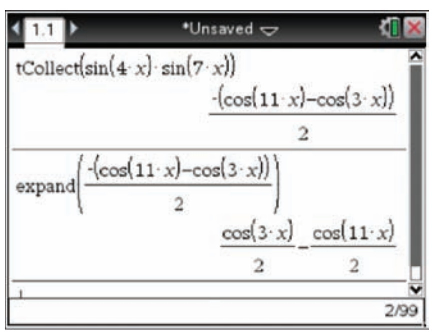
$=\frac{1}{2}[\cos(3x)-\cos(11x)]$

Write the result.

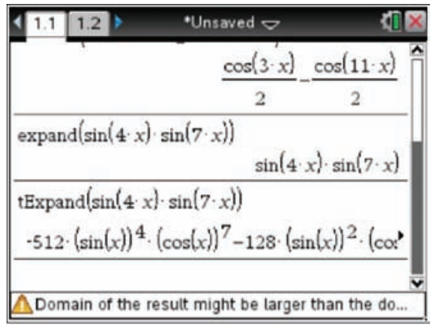
$\sin(4x)\sin(7x)=\frac{1}{2}[\cos(3x)-\cos(11x)]$

TI-Nspire CAS

Use tCollect() from the catalogue (k).
You can expand the terms using b, 3: Algebra, 3: Expand and using /v to insert the answer. The calculator will automatically insert the expression from the answer.



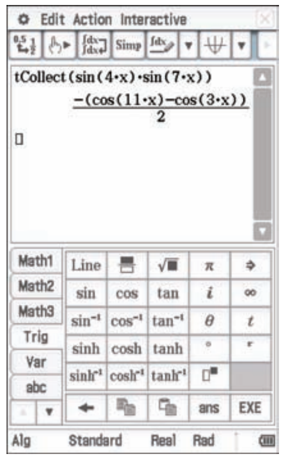
You cannot use expand() on its own or texpand() because you will then get an expression that has only powers of sin (x) and cos (x).



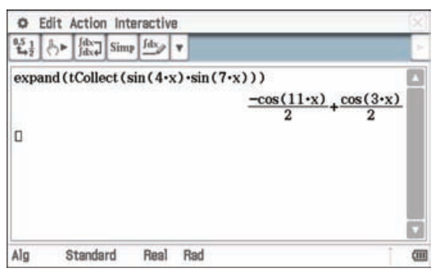
ClassPad

Use the Main $\sqrt{\alpha}$ application.
Tap **Interactive**, then **Transformation** and **tCollect**, type **tCollect** or choose it from the catalogue.

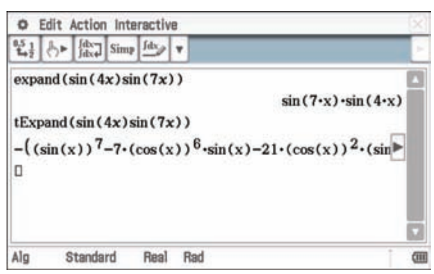
It is important that the 4***x*** and the 7***x*** are in brackets.
Note that the terms would need to be expanded.



This expansion can be done on the calculator by expand(tCollect(sin(4***x***)sin(7***x***))).



You cannot use **expand** or **tExpand** (also on the menu found by tapping **Action** then **Transformation**) on their own, as the screen on the right shows.
tExpand will use the double angle and the angle sum formulas to expand sin (4x) and sin (7x) until everything is expressed in terms of sin (x) and cos (x).



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EXERCISE 9.06 Products to sums

Concepts and techniques

1 Example 17 Given that $\sin(40°) \approx 0.6428$ and $\cos(40°) \approx 0.7660$, use an appropriate Products as sums and differences formula to evaluate the following expressions, giving your answers correct to 3 decimal places.

a $\cos(10°)\cos(50°)$
b $\sin(10°)\sin(50°)$
c $\sin(10°)\cos(50°)$
d $\cos(10°)\sin(50°)$

2 Example 18 Express the following as sums or differences of trigonometric functions.

a $2\sin(5x)\cos(7x)$
b $8\sin(3x)\sin(9x)$
c $6\cos(4x)\cos(6x)$
d $4\cos(5x)\sin(6x)$
e $\sin(6x)\cos(10x)$
f $3\cos(10x)\cos(14x)$
g $5\sin(2x)\sin(3x)$
h $7\cos(9x)\cos(11x)$
i $\cos(8x)\sin(x)$

3 Express each of the following as sums or differences of trigonometric functions and hence find their exact values.

a $\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right)$
b $\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right)$
c $\sin\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{4}\right)$
d $2\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{4}\right)$

Reasoning and communication

4 Example 16 Show that $\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

5 Show that $2\cos(A)\sin(B) = \sin(A+B) - \sin(A-B)$

6 Show that $\sin(g)\sin(f) = \frac{1}{2}[\cos(f-g)-\cos(f+g)]$

7 Show that $\sin(c+d) = 2\sin(c)\cos(d) - \sin(c-d)$

8 Show that $\sin(11x)\cos(7x) - \sin(8x)\cos(4x) = \sin(3x)\cos(15x)$.

9 Show that $4\sin(3x)\sin(5x)\sin(8x) = \sin(6x) + \sin(10x) - \sin(16x)$.

10 Waves breaking on a beach may have travelled thousands of kilometres from where they were formed by wind blowing on the water. The waves can actually consist of smaller waves of different frequencies, amplitudes and wavelengths because they may have arisen in different parts of the ocean. Suppose that the waves arriving at Kirra, Qld. on a particular day actually consist of waves that can be modelled as $\sin(5t)$, $\sin(6t)$, $\sin(7t)$ and $\sin(8t)$. Show that every sixth wave is larger.

Shutterstock.com/Joshua Rainey Photography

9.07 SUMS TO PRODUCTS

You have examined the shapes of the trigonometric functions. In practice, the functions rarely occur singly, so we need to be able to combine the functions to obtain the overall shape of a combined graph.

In AC circuits, the effects of inductors and capacitors are such that the voltages in circuits involving these elements consist of two parts that are out of phase by 90°. We know $\sin(\theta + 90°) = \cos(\theta)$, so this effectively means that there is a combination of the sine and cosine functions involved in the voltages of AC circuits.

However, the coefficients of the functions are different, since they are determined by the sizes of the circuit elements, the applied voltages and other properties of the circuit.

There is a practical need to be able to express a function of the form $a \sin(x) + b \cos(x)$ as a single function. You can do this by using the addition theorems in reverse.

Example 19

Express $3 \sin(x) + 5 \cos(x)$ in the form $A \sin(x + \theta)$ and sketch the graph for $x = 0$ to 2π.

Solution

Use the addition theorem.

$$A \sin(x + \theta) = A[\sin(x) \cos(\theta) + \cos(x) \sin(\theta)]$$
$$= A \cos(\theta) \sin(x) + A \sin(\theta) \cos(x)$$

Let $A \sin(x + \theta) = 3 \sin(x) + 5 \cos(x)$.

$$3 \sin(x) + 5 \cos(x) = A \cos(\theta) \sin(x) + A \sin(\theta) \cos(x)$$

Compare the LHS and RHS.

$A \cos(\theta) = 3$ and $A \sin(\theta) = 5$.

We know that $A^2 = A^2 \times 1$, so we can use $\cos^2(\theta) + \sin^2(\theta) = 1$ as follows.

$$A^2 = A^2[\cos^2(\theta) + \sin^2(\theta)]$$

Expand and factorise.

$$= [A \cos(\theta)]^2 + [A \sin(\theta)]^2$$

Substitute values.

$$= 3^2 + 5^2 = 34$$

Take square roots to evaluate A.

$$A = \sqrt{34} \approx 5.83$$

Use the definition for $\tan(\theta)$.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$= \frac{A \sin(\theta)}{A \cos(\theta)}$$
$$= \frac{5}{3}$$
$$\approx 1.67$$

Write the result.

$$3 \sin(x) + 5 \cos(x) = \sqrt{34} \sin\left[x + \tan^{-1}\left(\frac{5}{3}\right)\right]$$

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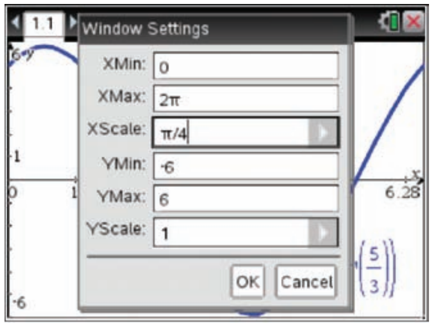
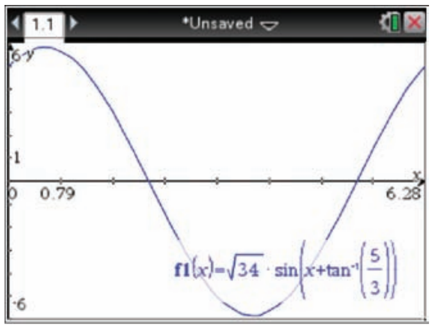
You can sketch this graph using a CAS calculator.

TI-Nspire CAS

Insert a graph and change the window settings to $0 \le x \le 2\pi$ and $-6 \le y \le 6$ using b, 4: Window/Zoom and 1: Window settings.



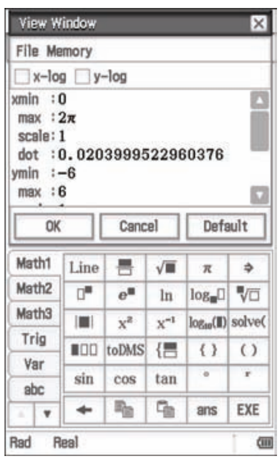
Enter the function as $f1(x) = \sqrt{34}\sin\left[x + \tan^{-1}\left(\frac{5}{3}\right)\right]$.



ClassPad

Use the **Graph&Table** application. Set the calculator to radian measure (**Rad**).

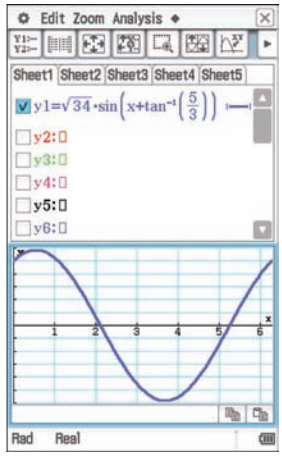
Tap and adjust the viewing window for $0 \le x \le 2\pi$ and $-6 \le y \le 6$, using a scale of 1.



Enter y1$=\sqrt{34}\sin(x+\tan^{-1}(\frac{5}{3}))$

Use the Trig menu to access the fraction and trigonometric functions. Press EXE to finish entering the function.

Tap [graph icon] to draw the graph.



You can use the change of variables $A = x + y$ and $B = x - y$ to express sums as products.

First, we show that

$$\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right].$$

Write the change of variables. $A = x + y$ and $B = x - y$

Solve as simultaneous equations for x and y. $x=\frac{1}{2}(A+B)$ and $y=\frac{1}{2}(A-B)$

Write the formula for $2\sin(x)\cos(y)$. $2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$

Substitute for x, y, $(x + y)$ and $(x - y)$. $2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]=\sin(A)+\sin(B)$

Reverse to get the result. $\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$

QED

The same change of variables can be used with the other formulas to give the following.

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IMPORTANT

Sums to products

The following identities apply for all values of A and B.

$$\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$$

$$\sin(A)-\sin(B)=2\cos\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$$

$$\cos(A)+\cos(B)=2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$$

$$\cos(A)-\cos(B)=-2\sin\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$$

Note: the *last* formula has a *negative* sign.

The sums to products rules can be used to analyse wave interference.

Example 20

A sound wave can be modelled as a sine or cosine function specifying the displacement of air molecules as a function of time. For a **frequency** of f Hz (hertz), the displacement can be written as $A\sin(2\pi ft)$, where t is in seconds and A is the maximum displacement. Find the resulting sound when two guitar strings are plucked at frequencies of 256 Hz and 258 Hz with similar loudness.

Solution

Add these together to obtain the resulting sound.

$$A\sin(2\pi f_1 t)+A\sin(2\pi f_2 t)$$

Factorise.

$$=A[\sin(2\pi f_1 t)+\sin(2\pi f_2 t)]$$

Use the sum-to-product rule.

$$=A\left\{2\sin\left[\frac{1}{2}(2\pi f_1 t+2\pi f_2 t)\right]\cos\left[\frac{1}{2}(2\pi f_1 t-2\pi f_2 t)\right]\right\}$$

Factorise.

$$=A\left\{2\sin\left[2\pi\left(\frac{f_1+f_2}{2}\right)t\right]\cos\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]\right\}$$

$$=2A\sin\left[2\pi\left(\frac{f_1+f_2}{2}\right)t\right]\cos\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]$$

Substitute for f_1 and f_2 and simplify.

$$=2A\sin(2\pi\times 257t)\cos(2\pi\times 1t)$$

Rearrange terms.

$$=2A\cos(2\pi\times 1t)\sin(2\pi\times 257t)$$

Let $f_3 = 257$ and $A' = 2A\cos(2\pi f_4 t)$ where $f_4 = 1$.

$$=A'\sin(2\pi f_3 t)$$

Compare the form of the last expression with the original expressions and write a conclusion.

The sound will have a frequency of 257 Hz, but the loudness will vary from loud to soft with a frequency of 1 Hz. This variation is called 'beating' and can be heard clearly when tuning the guitar.

You can use the sums to products formulas to simplify expressions containing exact values.

Example 21

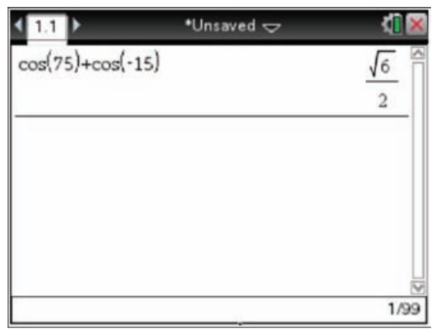
Simplify the expression cos (75°) + cos (–15°) and find its exact value.

Solution

Write the expression.	$\cos(75^\circ)+\cos(-15^\circ)$
Express cos (–15°) in terms of cos (15°).	$=\cos(75^\circ)+\cos(15^\circ)$
Use cos (*A*) + cos (*B*) = 2 cos (*A* + *B*) cos (*A* – *B*).	$=2\cos\left[\frac{1}{2}(75^\circ+15^\circ)\right]\cos\left[\frac{1}{2}(75^\circ-15^\circ)\right]$
Simplify.	$=2\cos(45^\circ)\cos(30^\circ)$
Substitute exact values.	$=2\times\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}$
Simplify.	$=\frac{\sqrt{6}}{2}$
Write the result.	$\cos(75^\circ)+\cos(-15^\circ)=\frac{\sqrt{6}}{2}$

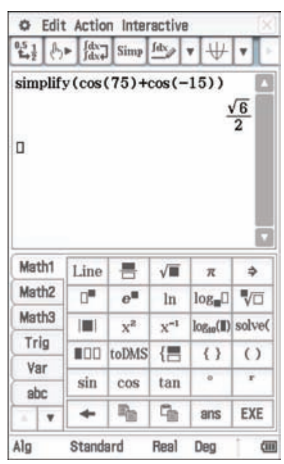
TI-Nspire CAS

Make sure that your calculator is set on degrees.



ClassPad

This question can be done using **simplify**.
Make sure that your calculator is set on **Standard** and **Deg**.
Use the Main $\sqrt{\alpha}$ application and tap **Action** then **Transformation** then **simplify**.



You can also use the sums to products formulas to prove trigonometric identities.

EXERCISE 9.07 Sums to products

Concepts and techniques

You should use a CAS calculator to sketch the graphs in questions **1** to **6**.

1 Example 19 **a** Express $6 \sin (x) + 8 \cos (x)$ in each of the following forms and sketch the graphs for $x = 0$ to 2π.

i $A \sin (x + \theta)$ **ii** $A \cos (x + \theta)$ **iii** $A \sin (x - \theta)$ **iv** $A \cos (x - \theta)$

b Comment on the graphs obtained.

2 Express $\sin(x)+\sqrt{3}\cos(x)$ in the form $A \sin (x + \theta)$ and sketch the graph for $x = 0$ to 2π.

3 Express $\sin (x) + \cos (x)$ in the form $A \cos (x - \theta)$ and sketch the graph for $x = 0$ to 2π.

4 Express $\sqrt{3}\sin(x)-\cos(x)$ in the form $A \cos (x + \theta)$ and sketch the graph for $x = 0$ to 2π.

5 Express $12 \sin (x) + 5 \cos (x)$ in the form $A \sin (x + \theta)$ and sketch the graph for $x = 0$ to 2π.

6 Express $8 \sin (x) - 15 \cos (x)$ in the form $A \sin (x - \theta)$ and sketch the graph for $x = 0$ to 2π.

7 Express each of the following as a product.

a $\sin (3x) + \sin (7x)$ **b** $\sin (8x) - \sin (2x)$ **c** $\cos (5x) + \cos (9x)$

d $\cos (11x) - \cos (7x)$ **e** $\sin (4x) + \sin (9x)$ **f** $\sin (2x) - \sin (5x)$

g $\cos (x) + \cos (5x)$ **h** $\cos (3x) - \cos (11x)$ **i** $\sin (x) + \sin (2x)$

8 Example 21 Use the formula $\cos(A)+\cos(B)=2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$ to simplify the expression $\cos (120°) + \cos (60°)$.

9 Use the formula $\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$ to simplify the expression $\sin (75°) + \sin (15°)$.

10 Use the formula $\sin(A)-\sin(B)=2\cos\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$ to simplify the expression $\sin (135°) - \sin (-45°)$.

Reasoning and communication

11 Show that $2 \cos (x) \cos (y) = \cos (x + y) + \cos (x - y)$.

12 Use the expansions of $\sin (2x)$, $\sin (x + y)$ and $\cos (x - y)$ to show that

$$\sin(A)+\sin(B)=2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right].$$

(*Hint*: Write $A = 2x$ and $B = 2y$.)

13 Use the expansions of $\cos (2x)$, $\cos (x + y)$ and $\cos (x - y)$ to show that

$$\cos(A)+\cos(B)=2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right].$$

14 Example 20 A 6th (lowest, E) string on a guitar is in tune at 82.4 Hz and is being tuned upward. When it is pushed against the 5th fret, it should give the same note (110 Hz) as the next string, A. In fact, when both strings are plucked, there is a 'beat' with intervals of 0.5 s. When the A

string is tightened, the 'beat' changes to every second. The correct note can be modelled as a sine function, with the formula $y = a\sin\left(\frac{110t}{2\pi}\right)$, where a is the loudness and t is time. Show that this is consistent with making the guitar closer to being in tune using a sums to products formula.

15 a Express $\sin(x + h) - \sin(x)$ as a product.

b Find an expression for $\frac{\sin(x+h)-\sin(x)}{h}$.

c Given that $\lim_{z\to 0}\frac{\sin(z)}{z}=1$, find $\frac{d}{dx}\sin(x)$ from the definition of the derivative.

9.08 TRIGONOMETRIC IDENTITIES

The use of trigonometric functions extends far beyond the solution of geometric problems involving triangles. In fact, their most important application is in the modelling of periodic phenomena. These applications include light, sound, electrical circuits, electronics, shock absorbers and even the way a door with a damper closes. While it is obvious that AC electrical theory uses trigonometric functions, it is not so obvious that advanced DC current theory also uses them. However, to model what happens when a switch is turned on or off, trigonometric functions are vital in the analysis of transient currents that can easily damage sensitive components in a circuit.

When you prove identities, start with the side that is more complicated.

Example 22

Prove the identity $2\sin^2(\theta)\cos^2(\theta)=\frac{1}{2}\sin^2(2\theta)$

Solution

Start with the RHS.	$\frac{1}{2}\sin^2(2\theta)$
Use the double angle formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.	$=\frac{1}{2}[\sin(2\theta)]^2$ $=\frac{1}{2}[2\sin(\theta)\cos(\theta)]^2$
Simplify.	$=\frac{1}{2}\times 4\sin^2(\theta)\cos^2(\theta)$ $= 2\sin^2(\theta)\cos^2(\theta)$
Compare with the LHS.	$=$ LHS
Write the result.	$2\sin^2(\theta)\cos^2(\theta)=\frac{1}{2}\sin^2(2\theta)$ **QED**

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Sometimes it is easier to simplify both the LHS and the RHS.

Example 23

Prove the identity $\dfrac{1}{\operatorname{cosec}(x)+\cot(x)}=\dfrac{\frac{1}{2}[1-\cos(2x)]}{\tan(x)\cos(x)+\frac{1}{2}\sin(2x)}$

Solution

Start with the LHS by simplifying the fraction.

$$\frac{1}{\operatorname{cosec}(x)+\cot(x)}$$

Use the reciprocal identities $\operatorname{cosec}(x)=\dfrac{1}{\sin(x)}$ and $\cot(x)=\dfrac{1}{\tan(x)}$.

$$=\frac{1}{\frac{1}{\sin(x)}+\frac{1}{\tan(x)}}$$

Simplify using the identity $\tan(x)=\dfrac{\sin(x)}{\cos(x)}$.

$$=\frac{1}{\frac{1}{\sin(x)}+\frac{\cos(x)}{\sin(x)}}$$

Express and simplify using a common denominator.

$$=\frac{\sin(x)}{1+\cos(x)}$$

Write the result for the LHS.

$$\frac{1}{\operatorname{cosec}(x)+\cot(x)}=\frac{\sin(x)}{1+\cos(x)}$$

Now simplify the RHS.
Use the double angle formulas $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = 1 - 2\sin^2(x)$.

$$\frac{\frac{1}{2}[1-\cos(2x)]}{\tan(x)\cos(x)+\frac{1}{2}\sin(2x)}$$

$$=\frac{\frac{1}{2}\left(1-\left[1-2\sin^2(x)\right]\right)}{\frac{\sin(x)}{\cos(x)}\cos(x)+\frac{1}{2}\times 2\sin(x)\cos(x)}$$

Cancel and collect terms.

$$=\frac{\sin^2(x)}{\sin(x)+\sin(x)\cos(x)}$$

Take out $\sin(x)$ as a common factor and cancel.

$$=\frac{\sin^2(x)}{\sin(x)[1+\cos(x)]}$$

$$=\frac{\sin(x)}{1+\cos(x)}$$

$$=\text{LHS}$$

Write the result.

$$\frac{1}{\operatorname{cosec}(x)+\cot(x)}=\frac{\frac{1}{2}[1-\cos(2x)]}{\tan(x)\cos(x)+\frac{1}{2}\sin(2x)}$$

You can simplify trigonometric expressions in terms of either sine or cosine only.

Example 24

WS Calculating exact values of trigonometric functions

Rewrite the expression 3 sin (2x) cos (x) + sin (x) cos (2x) in terms of sin (x) only.

Solution

Use the double angle formulas sin (2x) = 2 sin (x) cos (x) and cos (2x) = 1 – 2 $\sin^2(x)$ to simplify the expression.	$3\sin(2x)\cos(x) + \sin(x)\cos(2x)$ $= 3 \times 2\sin(x)\cos(x)\cos(x) + \sin(x) \times [1 - 2\sin^2(x)]$
Collect terms.	$= 6\sin(x)\cos^2(x) + \sin(x) - 2\sin^3(x)$
Use the rearrangement of the Pythagorean identity $1 - \sin^2(x) = \cos^2(x)$	$= 6\sin(x)[1 - \sin^2(x)] + \sin(x) - 2\sin^3(x)$
Expand the brackets and collect terms.	$= 6\sin(x) - 6\sin^3(x) + \sin(x) - 2\sin^3(x)$ $= 7\sin(x) - 8\sin^3(x)$
Write the result.	$3\sin(2x)\cos(x) + \sin(x)\cos(2x) = 7\sin(x) - 8\sin^3(x)$

EXERCISE 9.08 Trigonometric identities

Reasoning and communication

1 Example 22 Prove the identity $\frac{1}{8}\sin^2(4x) = \frac{1}{2}\sin^2(2x)\cos^2(2x)$

2 Show that $2\sin^2(6x) = 8[\sin^2(3x) - \sin^4(3x)]$

3 Prove the following identities.

a $2\sin(x) - 2\sin^2(x) - \cos^2(x) = -[\sin(x) - 1]^2$

b $\frac{1}{2}\left[\cos(x) + \cos\left(\frac{\pi}{3}\right)\right] = \cos\left(\frac{x}{2} + \frac{\pi}{6}\right)\cos\left(\frac{x}{2} - \frac{\pi}{6}\right)$

4 Example 23 Prove the following identities.

a $\frac{-1}{\operatorname{cosec}(x) + \cot(x)} = \frac{\frac{1}{2}[1 - \cos(2x)]}{\tan(x)\cos(x) + \frac{1}{2}\sin(2x)}$

b $\frac{1}{4}[\sin(x + y) + \sin(x - y)]^2 = [\sin(x) - \sin(x)\sin(y)]^2 - 2\sin^2(x)\sin^2(y) + 2\sin^2(x)\sin(y)$

c $\frac{1}{\sin(x) + \operatorname{cosec}(x)} = \frac{\frac{1}{2}\sec(x)\sin(2x)}{1 - \cos^2(x) + \operatorname{cosec}^2(x) - \cot^2(x)}$

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5 Example 24 Rewrite the expression $\frac{1}{3}\sin(3x)+\cos(2x)$ in terms of sin only.

6 Rewrite the expression $-2\sin^2(x)\cos(2x)$ in terms of $\cos(x)$ only.

7 Prove that $\sec^2(x) = \sec(x)\operatorname{cosec}(x)\tan(x)$.

8 Prove that $\frac{1}{\operatorname{cosec}(x)+1}+\frac{1}{\operatorname{cosec}(x)-1}=2\tan(x)\sec(x)$.

9 Show that $1 + \cos(2x) = 2\cos^2(x)$.

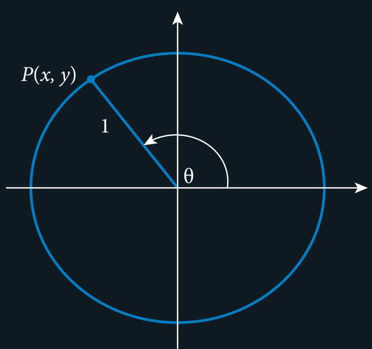
10 Prove that $\cot(x)+\cot(y)=\frac{\operatorname{cosec}(x)\operatorname{cosec}(y)}{\operatorname{cosec}(x+y)}$.

9 CHAPTER SUMMARY

TRIGONOMETRIC IDENTITIES

- Trigonometric functions. For a point $P(x, y)$ at an angle θ on the unit circle,

$\sin(\theta) = y$, $\cos(\theta) = x$ and $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$.



- Compound angles are found in expressions that involve more than one angle, such as $\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)$ or $\cos(2\theta)$.
- An identity is a formula that is true for every value.
- Reciprocal trigonometric relationships are

$$\operatorname{cosec}(x)=\frac{1}{\sin(x)},\ \sec(x)=\frac{1}{\cos(x)} \text{ and}$$

$$\cot(x)=\frac{1}{\tan(x)}.$$

- The symmetry of the unit circle gives:

2nd quadrant

- $\operatorname{cosec}(\pi - \theta) = \operatorname{cosec}(\theta)$
- $\sec(\pi - \theta) = -\sec(\theta)$
- $\cot(\pi - \theta) = -\cot(\theta)$

3rd quadrant

- $\operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}(\theta)$
- $\sec(\pi + \theta) = -\sec(\theta)$
- $\cot(\pi + \theta) = \cot(\theta)$

4th quadrant

- $\operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec}(\theta) = \operatorname{cosec}(-\theta)$
- $\sec(2\pi - \theta) = \sec(\theta) = \sec(-\theta)$
- $\cot(2\pi - \theta) = -\cot(\theta) = \cot(-\theta)$

- The Pythagorean identities are:

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta)$

Angle sum and difference identities are:

- $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
- $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
- $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
- $\tan(A+B)=\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$
- $\tan(A-B)=\frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$

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Complementary function identities are:

- $\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos(\theta)$
- $\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin(\theta)$
- $\tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot(\theta)$
- $\sin\left(\frac{3\pi}{2} \pm \theta\right) = -\cos(\theta)$
- $\cos\left(\frac{3\pi}{2} \pm \theta\right) = \pm\sin(\theta)$
- $\tan\left(\frac{3\pi}{2} \pm \theta\right) = \mp\cot(\theta)$

Double angle formulas are:

- $\sin(2A) = 2\sin(A)\cos(A)$
- $\cos(2A) = \cos^2(A) - \sin^2(A)$
  $= 2\cos^2(A) - 1$
  $= 1 - 2\sin^2(A)$
- $\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$

Products as sums and differences formulas are:

- $\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$
- $\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$
- $\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$
- $\cos(A)\sin(B) = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

Sums to products formulas are:

- $\sin(A) + \sin(B) = 2\sin\left[\frac{1}{2}(A + B)\right]\cos\left[\frac{1}{2}(A - B)\right]$
- $\sin(A) - \sin(B) = 2\cos\left[\frac{1}{2}(A + B)\right]\sin\left[\frac{1}{2}(A - B)\right]$
- $\cos(A) + \cos(B) = 2\cos\left[\frac{1}{2}(A + B)\right]\cos\left[\frac{1}{2}(A - B)\right]$
- $\cos(A) - \cos(B) = -2\sin\left[\frac{1}{2}(A + B)\right]\sin\left[\frac{1}{2}(A - B)\right]$

9

CHAPTER REVIEW
TRIGONOMETRIC IDENTITIES

Multiple choice

1 Example 1 $\sin(60°) =$

A $\frac{2}{\sqrt{3}}$ B $\frac{3}{\sqrt{3}}$ C $\frac{\sqrt{3}}{2}$ D $\frac{1}{\sqrt{3}}$ E $\frac{1}{2}$

2 Example 1 $\text{cosec}\left(\frac{\pi}{3}\right) =$

A $\frac{2}{\sqrt{3}}$ B $\frac{3}{\sqrt{3}}$ C $\frac{\sqrt{3}}{2}$ D $\frac{1}{\sqrt{3}}$ E $\frac{1}{2}$

3 Example 2 $\sin\left(\pi - \frac{\pi}{6}\right) =$

A $\frac{2}{\sqrt{3}}$ B $\frac{3}{\sqrt{3}}$ C $\frac{\sqrt{3}}{2}$ D $\frac{1}{\sqrt{3}}$ E $\frac{1}{2}$

4 Example 2 $\tan(360° - 30°) =$

A $\tan(30°)$ B $-\tan(60°)$ C $-\tan(30°)$ D $\cot(30°)$ E $\tan(60°)$

5 Example 13 $\cos(75°) =$

A $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ B $\frac{\sqrt{6}}{4}$ C $\frac{3\sqrt{2}}{4}$ D $\frac{\sqrt{6}+\sqrt{2}}{4}$ E $\frac{\sqrt{3}}{4} - \frac{\sqrt{2}}{4}$

6 Example 15 $\sin\left(\frac{5\pi}{8}\right) =$

A $\frac{\sqrt{2+\sqrt{2}}}{4}$ B $\frac{\sqrt{2-\sqrt{2}}}{2}$ C $\frac{2+\sqrt{2}}{4}$ D $\frac{\sqrt{2-\sqrt{2}}}{4}$ E $\frac{\sqrt{2+\sqrt{2}}}{2}$

7 Example 18 $\cos(5x)\cos(9x) =$

A $\frac{1}{2}[\cos(7x) + \cos(2x)]$

B $\frac{1}{2}[\sin(14x) + \sin(4x)]$

C $2[\cos(7x) - \cos(2x)]$

D $\frac{1}{2}[\sin(7x) + \sin(2x)]$

E $\frac{1}{2}[\cos(14x) + \cos(4x)]$

8 Example 21 $\sin(12x) - \sin(8x) =$

A $\frac{1}{2}\sin(20x)\cos(4x)$

B $2\cos(10x)\sin(2x)$

C $2\sin(10x)\cos(2x)$

D $2\sin(20x)\cos(4x)$

E $\frac{1}{2}[\cos(2x)\cos(10x)]$

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Short answer

9 Example 2 Write a simplified expression for:

a $\sec(2\pi - \theta)$ b $\operatorname{cosec}(\pi - \theta)$ c $\cot(2\pi - \theta)$

d $\sec(\pi + \theta)$ e $\cot(\pi + \theta)$ f $\operatorname{cosec}(2\pi - \theta)$

10 Example 8 Write a simplified expression for:

a $\sec\left(\frac{\pi}{2} - \theta\right)$ b $\operatorname{cosec}\left(\frac{3\pi}{2} - \theta\right)$ c $\cot\left(\frac{\pi}{2} - \theta\right)$

d $\sec\left(\frac{3\pi}{2} + \theta\right)$ e $\cot\left(\frac{3\pi}{2} - \theta\right)$ f $\operatorname{cosec}\left(\frac{\pi}{2} - \theta\right)$

11 Example 3 Sketch the graph of $y = 3\operatorname{cosec}(x)$ for $0 \le x \le 2\pi$.

12 Example 3 Sketch the graph of $y = 2\sec(x) - 1$ for $0 \le x \le 2\pi$.

13 Example 5 Prove the identity $\sec^2(x) - 1 = \tan^3(x)\cot(x)$

14 Example 14 Find the exact value of $\cos\left(\frac{\pi}{12}\right)$.

15 Example 11 Find the exact value of $\tan\left(\frac{\pi}{12}\right)$.

16 Example 15 Find the exact value of $\sin\left(\frac{7\pi}{12}\right)$.

Application

17 Find an expression for $\sin(4x)$ in terms of $\cos(x)$ and $\sin(x)$ only.

18 Show that $\sin(A)\sin(B) = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$.

19 Show that $\dfrac{\cos(x) + \sec(x)}{\cot(2x)} = \dfrac{2\tan(x)\cdot[\cos^2(x) + 1]}{\cos(x)\cdot[1 - \tan^2(x)]}$

20 Show that $\dfrac{1 - \cos(x)}{8} = [4\sin(x)\cos^3(x) - 2\sin(x)\cos(x)]^2$

Practice quiz

21 Show that $\cos(x)\tan(x)\sin(3x) = \sin^2(x)\cos(2x) + \sin(x)\cos(x)\sin(2x)$

22 Rewrite the expression $3\sin(2x)\sin(x)\cos(x) + \cos(x)\cos(2x)$ in terms of $\cos(x)$ only.